MODEL INDEPENDENT $f_0(500)$ AND
$f_0(980)$ MESON PARAMETERS BY PION
SCALAR FF ANALYSIS

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The pion scalar FF $\Gamma_\pi(t)$ is defined by the matrix element of the scalar quark density

$$<\pi^i(p_2) | \bar{m}(\bar{u}u + \bar{d}d) | \pi^j(p_1) > = \delta^{ij} \Gamma_\pi(t)$$

where $t = (p_2 - p_1)^2$ and $\bar{m} = \frac{1}{2}(m_u + m_d)$. 
Properties of $\Gamma_\pi(t)$:

They are similar to the properties of the pion EM FF

• it is analytic in the whole complex $t$-plane besides for a cut along the positive real axis starting at $t = 4m^2_\pi$

• for real values $t < 4m^2_\pi$ $\Gamma_\pi(t)$ is real
  
  $\Rightarrow$ it implies the so-called reality condition

  $$\Gamma^*_\pi(t) = \Gamma_\pi(t^*)$$

• at $t = 0$ $\Gamma_\pi(t)$ coincides with the pion sigma term

  however, in our considerations we normalize it to one

  $\Gamma_\pi(0) = 1$
• if $\Gamma_\pi(t)$ - on the upper boundary of the cut

  $\Rightarrow$ the unitarity condition is obeyed by $\Gamma_\pi(t)$

  $Im \Gamma_\pi(t) = \langle \pi(p')\pi(p) \mid T \mid n > < n \mid \bar{m}(\bar{u}u + \bar{d}d) \mid 0 >$

  where the sum runs over a complete set of allowed states like $2\pi, 4\pi, ... K\bar{K}$, etc., which create additional branch cuts on the positive real axis of the $t$-plane between $4m^2_\pi$ and $\infty$.

• in the elastic region $4m^2_\pi \leq t \leq 16m^2_\pi$ only the first term in the unitarity condition contributes

  $\Rightarrow Im \Gamma_\pi(t) = M^0_0 \Gamma^*_\pi$ where $M^0_0$ is $I = J = 0$ partial wave $\pi\pi$ scattering amplitude

  $M^0_0 = e^{i\delta^0_s} sin\delta^0_s$

  with $\delta^0_s$ the $S$-wave isoscalar $\pi\pi$ phase shift
• ⇒ the elastic unitarity condition for $\Gamma_\pi(t)$ is

$$Im\Gamma_\pi(t) = e^{i\delta_0}sin\delta_0\Gamma_\pi^*$$

from where the identity

$$\delta_\pi \equiv \delta_0^0$$

follows, where $\delta_\pi$ is phase of the pion scalar FF

**NOTE:**

This identity - revealed to be valid well above $t = 4m_K^2 \approx 1GeV^2$, where the inelastic two-body channel $\pi\pi \rightarrow K\bar{K}$ is opened - is crucial as it enables us to find an explicit form of the pion scalar FF

• The asymptotic behavior of $\Gamma_\pi(t)$ is

$$\Gamma_\pi(t)|_{|t|\rightarrow\infty} \sim 1/t$$
Then, taking into account these properties

by application of the Cauchy formula

the dispersion relation without subtractions

\[
\Gamma_\pi(t) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{Im \Gamma_\pi(t')}{t' - t} dt'
\]

or the dispersion relation with one subtraction at \(t = 0\)

\[
\Gamma_\pi(t) = 1 + \frac{t}{\pi} \int_{4m_\pi^2}^{\infty} \frac{Im \Gamma_\pi(t')}{t'(t' - t)} dt'
\]

or two subtractions etc.

can be derived.
These dispersion relations,

together with the elastic unitarity condition
give

the Omnes-Muskelishvili integral equations

and their solutions are just the phase representations of $\Gamma_\pi(t)$

$$
\Gamma_\pi(t) = P_n(t) \exp \left[ \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\delta_0^0(t')}{t' - t} \, dt' \right]
$$

or

$$
\Gamma_\pi(t) = P_n(t) \exp \left[ \frac{t}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\delta_0^0(t')}{t'(t' - t)} \, dt' \right]
$$

etc.

with arbitrary, however at $t = 0$ normalized,

polynomial $P_n(t)$.  


So, if the S-wave iso-scalar $\pi\pi$-scattering phase shift $\delta_0^0$ is known, one can find explicit form of the scalar pion FF $\Gamma_\pi(t)$ and its poles $f_0(500)$ and $f_0(980)$.

We have available data on $\delta_0^0$ (see Fig.1)
Now we explain "what we understand under the model independent approach".

We describe existing data on $\delta^0_0$ in a fully model independent way.

With this aim we carry out the conformal mapping

$$q = [(t - 4)/4]^{1/2} \quad m_\pi = 1$$

of the two sheeted Riemann surface on which $\Gamma_\pi(t)$ is defined onto the pion c.m. momentum $q$-plane.

⇒ there are only poles and zeros in $q$-plane and one can represent the pion scalar FF in the form of Pad’e type approximation

$$\Gamma_\pi(t) = \frac{\sum_{n=0}^M a_n q^n}{\prod_{i=1}^N (q-q_i)}$$
Poles $q_i$ can be always found to be placed:

- on the **imaginary axis** or

- two of them symmetrically according to it

As the $\Gamma_\pi(t)$ is real analytic function

$\Rightarrow$ **coefficients** $a_n$ with $M$ even (odd) are real (pure imaginary), respectively.

Then, taking into account the **threshold behavior** of $\delta^0_0$ one can derive from all previous its $q$-dependence in the form

$$
\delta^0_0(t) = \arctan \frac{A_1q + A_3q^3 + A_5q^5 + A_7q^7 + \ldots}{1 + A_2q^2 + A_4q^4 + A_6q^6 + \ldots} \quad (\ast)
$$

with **new pure real coefficients** $A_i$, where $A_1$ is the $S$-wave iso-scalar $\pi\pi$ scattering length $a^0_0$. 


The **fitting procedure** has been carried out step by step.

First, **coefficients** $A_1$ and $A_2$ have been considered to be nonzero.

Then $A_1$, $A_2$ and $A_3$. etc.

Finally, we have found that $[5/4]$ Pad’e type approximation

\[
\delta_0^0(t) = \arctan \frac{A_1 q + A_3 q^2 + A_5 q^5}{1 + A_2 q^2 + A_4 q^4}
\]

with the values of parameters

\[
A_1 = 0.2351 \pm 0.0107; A_3 = 0.2706 \pm 0.0162; A_5 = -0.0248 \pm 0.0007
\]
\[
A_2 = 0.2137 \pm 0.0283; A_4 = -0.0443 \pm 0.0048
\]

and $\chi^2/ndf = 86/61 = 1.41$

gives the **best description** of $\delta_0^0(t)$ as it is presented by **full curve in Fig. 1**.
From the previous result of the fit one can see immediately that

\[ \lim_{q \to \infty} \delta_0^0(t) = \pi/2, \]

\( \Rightarrow \) one has to consider the phase representation with one subtraction.

We substitute for \( \delta_0^0(t) \) the expression obtained by the best fit, however, from the practical point of view of the calculations the equivalent mathematical relation

\[ \arctan z = \frac{1}{2i} \ln \frac{1+iz}{1-iz} \]

is used

\( \Rightarrow \)

\[ \Gamma_\pi(q) = P_n(q) \exp \frac{q^2 + 1}{\pi i} \int_0^\infty \frac{q' \ln \left( \frac{1+A_2q'^2+A_4q'^4+i(A_1q'+A_3q'^3+A_5q'^5)}{(1+A_2q'^2+A_4q'^4)-i(A_1q'+A_3q'^3+A_5q'^5)} \right)}{(q'^2 + 1)(q'^2 - q^2)} dq'. \]
As the integrand $\phi(q', q)$ in the previous expression is invariant under the transformation

$$q' \rightarrow -q'$$

i.e. it is even function of $q'$,

the integral can be changed into the following form

$$\Gamma_\pi(q) = P_n(q) \exp \frac{q^2 + 1}{2\pi i} \int_{-\infty}^{\infty} \frac{q' \ln \left( \frac{1 + A_2 q'^2 + A_4 q'^4}{1 + A_2 q^2 + A_4 q^4} + i (A_1 q' + A_3 q'^3 + A_5 q'^5) - i (A_1 q' + A_3 q'^3 + A_5 q'^5) \right)}{(q'^2 + 1)(q'^2 - q^2)} dq'$$

to be suitable for calculation by means of the theory of residues.

In order to carry out this programm one needs to know roots of polynomials in numerator and denominator under the logarithm, which generate branch points in $q$-plane.
NOTE:

The roots of denominator are complex conjugate roots of numerator!

The roots of numerator under the logarithm of integrand $\phi(q', q)$ are

\[
q_1 = -i1.8633297 \\
q_2 = -3.5830748 + i0.2832535 \\
q_3 = -1.3328447 + i1.2800184 \\
q_4 = 3.5830748 + i0.2832535 \\
q_5 = 1.3328447 + i1.2800184
\]
and the following **relations between roots of the denominator and the numerator** can be found

\[
\begin{align*}
q_1^* &= -q_1 \\
q_2^* &= -q_4 \\
q_3^* &= -q_5 \\
q_4^* &= -q_2 \\
q_5^* &= -q_3
\end{align*}
\]

Then the integral in \( \Gamma_\pi(t) \), considering the case \( q^2 < 0 \)
i.e. \( q = i \sqrt{\frac{4-t}{4}} \equiv ib \), takes the form

\[
I = \int_{-\infty}^{\infty} \frac{q' \ln \left( \frac{(q'-q_1)(q'-q_2)(q'-q_3)(q'-q_4)(q'-q_5)}{(q'-q_1^*)(q'-q_2^*)(q'-q_3^*)(q'-q_4^*)(q'-q_5^*)} \right)}{(q' + i)(q' - i)(q' + ib)(q' - ib)} dq'
\]

with all **singularities of its integrant** presented in Fig. 2.
Figure 2: Poles (×) and branch points (●) of the integrand $\phi(q', q)$ with the contour of integration in the upper half-plane.
Further it is convenient to split the integral into sum of two integrals

\[ I = \int_{-\infty}^{\infty} \frac{q' \ln(q' - q_2)(q' - q_3)(q' - q_4)(q' - q_5)}{(q' - q_1^*)} dq' + \]

\[ + \int_{-\infty}^{\infty} \frac{q' \ln(q' - q_2)(q' - q_3)(q' - q_4)(q' - q_5)}{(q' + i)(q' - i)(q' + ib)(q' - ib)} dq' = I_1 + I_2 \]

according to singularities to be placed in the upper half-plane or in the lower half-plane, respectively.

Let us start to calculate the first integral by the theory of residues

\[ \int \frac{q' \ln(q' - q_2)(q' - q_3)(q' - q_4)(q' - q_5)}{(q' + i)(q' - i)(q' + ib)(q' - ib)} dq' = 2\pi i \sum_n \text{Res}_n \]

where the contour of integration is closed in the upper half-plane (see Fig. 2).
As the **integral on the half-circle** is 0

\[ \Rightarrow \]

\[ I_1 = \int_{-\infty}^{\infty} \phi_1(q', q) dq' = 2\pi i \sum_n \text{Res}_n [- f_1 + f_2 + f_3 + f_4 + f_5] \]

Calculating **contributions of all cuts in upper half-plane** and **residua at the corresponding poles** one gets the result

\[ I_1 = + \frac{1}{2} \frac{2\pi i}{(q^2+1)} \ln \frac{(q+q_1^*)}{(i+q_2)(i+q_3)(i+q_4)(i+q_5)} \]

Similarly one can calculate also the **second integral**

\[ I_2 = \int_{-\infty}^{\infty} \phi_2(q', q) dq' = 2\pi i \sum_n \text{Res}_n [+ f_1 - f_2^* - f_3^* - f_4^* - f_5^*] \]

where the **contour of integration has to be closed in the lower half-plane**.
The **final result** for the integral \( I \) is

\[
I = I_1 + I_2 = \ln \frac{(q-q_1)}{(q+q_2)(q+q_3)(q+q_4)(q+q_5)} \frac{(i+q_2)(i+q_3)(i+q_4)(i+q_5)}{(i-q_1)}
\]

Substituting this expression into the

**pion scalar FF phase representation with one subtraction**

one obtains an **explicit form for the pion scalar FF** \( \Gamma_\pi(t) \)

\[
\Gamma_\pi(t) = P_n(t) \frac{(q-q_1)}{(q+q_2)(q+q_3)(q+q_4)(q+q_5)} \frac{(i+q_2)(i+q_3)(i+q_4)(i+q_5)}{(i-q_1)}
\]

in which the \(-q_3\) and \(-q_2\) **poles on the second Riemann sheet of \( t \)-variable** correspond to \( f_0(500) \) and \( f_0(980) \) scalar mesons, respectively, from where their parameters with errors

\[
\begin{align*}
    m_{f_0(500)} &= (360 \pm 33)\text{MeV} & \Gamma_{f_0(500)} &= (587 \pm 85)\text{MeV} \\
    m_{f_0(980)} &= (957 \pm 72)\text{MeV} & \Gamma_{f_0(980)} &= (164 \pm 142)\text{MeV}
\end{align*}
\]

are determined.
CONCLUSIONS

• new method for a prediction of the pion scalar FF behavior in elastic region has been developed

• by using as an input only the experimental data on S-wave iso-scalar $\pi\pi$ phase shift the parameters of the $f_0(500)$ and $f_0(980)$ scalar resonances are determined in a model independent way.