Constraints on QCD order parameters from $\eta \rightarrow \pi^+\pi^-\pi^0$

Marián Kolesár

(in collaboration with J. Novotný)

A) Resummed approach to $\chi$PT
B) The $\eta \rightarrow \pi^+\pi^-\pi^0$ decay width
C) Statistical analysis
D) Assumptions
E) Results - a first look
F) Summary, discussion and outlook

Hadron Structure'13, Tatranské Matliare, June 30, 2013
A) Quick introduction to resummed $\chi$PT

**Standard chiral perturbation theory ($N_f=3$) (Gasser, Leutwyler 1985)**

Generating functional:

$$e^{iZ_{eff}[\pi,v,a,s,p]} = \int D\pi \ e^{i \int d^4x \ L_{eff}[\pi,v,a,s,p]}$$

**SB$\chi$S: $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$** expansion in momenta and quark masses

**Building blocks:** $\pi^a \sim \pi, K, \eta$

$$U(x) = \exp \frac{i}{F_0} \pi^a(x)\lambda^a, \ M = \text{diag}(m_u, m_d, m_s)$$

**Effective Lagrangian:**

$$L_{eff} = L^{(2)} + L^{(4)} + L^{(6)} + \ldots$$

$$L^{(2(k+l))} \sim p^{2k} \chi^l, \ \chi = 2B_0 M$$

$$L^{(2)} = \frac{F_0^2}{4} \text{Tr}[D_\mu U D^\mu U^+] + (U^+\chi + \chi^+U)$$

$$L^{(4)} = \mathcal{L}^{(4)}(L_1 \ldots L_{10}) + \mathcal{L}^{(4)}_{WZ}$$

$$L^{(6)} = \mathcal{L}^{(6)}(C_1 \ldots C_{90}) + \mathcal{L}^{(6)}_{WZ}(C_1^W \ldots C_{23}^W)$$
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Resummed $\chi$PT - a special treatment of the chiral expansion

(Descotes-Genon, Fuchs, Girlanda, Stern 2004)

Motivation and aim
- possibly slow or irregular convergence of 3 flavour chiral series
- Standard $\chi$PT implicitly assumes good convergence, hides uncertainties
- express these assumptions in terms of parameters and uncertainty bands

Summary of the method
- Standard $\chi$PT Lagrangian and power counting
- only expansions derived directly from the generating functional trusted
- manipulations done in non-perturbative algebraic way
- explicitly to NLO, formally to all orders - collected in remainders
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A) Treatment of low energy constants

\[ \mathcal{O}(p^2): \quad F_0 - \text{pseudoscalar decay constant in the chiral limit} \]

\[ \Sigma - \text{quark condensate in the chiral limit} \quad (\Sigma = B_0 F_0^2) \]

\[ r - \text{strange to light quark ratio} \]

\[ R - \text{isospin violation} \quad (\text{light quark mass difference}) \]

\[
Z = \frac{F_0^2}{F_\pi^2}, \quad X = \frac{2 \hat{m} \Sigma}{F_\pi^2 M_\pi^2}, \quad r = \frac{m_s}{\hat{m}}, \quad R = \frac{(m_s - \hat{m})}{(m_d - m_u)}
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where \( \hat{m} = (m_u + m_d)/2 \)

\[ \mathcal{O}(p^4): \quad L_4-L_8 - \text{in terms of } F_P^2, M_P^2 \]

- algebraically, indirect remainders generated

\[ \mathcal{O}(p^6) \text{ and higher: } C_i \text{'s etc.} \]

Allowed range: \( X, Z \in (0,1) \) - only weak constraints from \( \pi\pi \) and \( \pi K \)

Two flavour values: \( X(2) = 0.81 \pm 0.07, \ Z(2) = 0.89 \pm 0.03 \)

Paramagnetic inequality: \( X < X(2), \ Z < Z(2) \)

Standard assumption \( Z \sim 1, \ X \sim 1, \ r \sim 25; \) latest \( S_\chi \text{PT fit } Z \sim 0.5, \ X \leq Z \)
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control of SB$\chi$S scenario

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(Bijnens, Jemos 2011)
B) The $\eta \rightarrow \pi^+ \pi^- \pi^0$ decay width

**PDG:** \[ \Gamma_{\text{exp}} = 296 \pm 16 \text{ eV} \]  \hspace{1cm} (PDG 2012)

The decay amplitude in terms of 4-point Green functions:

\[ F_\pi^3 F_\eta A(s, t, u) = G_{+-83}^{(4)} - \varepsilon_\pi G_{+-33}^{(4)} + \varepsilon_\eta G_{+-88}^{(4)} + \Delta_{G_D}^{(6)} \]

- to first order in isospin breaking, EM effects neglected
- physical mixing angles to all chiral orders and first in $1/R$

**Direct remainder expansion** around the Dalitz plot center

\[ \Delta_{G_D} = \Delta_A + \Delta_B (s - s_0) + \Delta_C (s - s_0)^2 + \Delta_D [(t - s_0)^2 + (u - s_0)^2] \]

19 parameters:
- LO: $X, Z, r, R$
- NLO: $L_1, L_2, L_3$
- direct rem.: $\Delta_A, \Delta_B, \Delta_C, \Delta_D$
- indirect rem.: $\Delta_{M_\pi}, \Delta_{F_\pi}, \Delta_{M_K}, \Delta_{F_K}, \Delta_{M_\eta}, \Delta_{F_\eta}, \Delta_{M_{38}}, \Delta_{Z_{38}}$
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$$\Delta_{GD} = \Delta_A + \Delta_B (s - s_0) + \Delta_C (s - s_0)^2 + \Delta_D [(t - s_0)^2 + (u - s_0)^2]$$

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- NLO: $L_1$, $L_2$, $L_3$
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C) Statistical analysis

Bayes’ theorem
(Stern et al. 2004)

\[
P(X_i | \Gamma_{\text{exp}}) = \frac{P(\Gamma_{\text{exp}} | X_i)P(X_i)}{\int dX_i P(\Gamma_{\text{exp}} | X_i)P(X_i)}
\]

- \(P(X_i | \Gamma_{\text{exp}})\) - probability density of \(X_i\) being true given \(\Gamma_{\text{exp}}\)

\[
P(\Gamma_{\text{exp}} | X_i) = \frac{1}{\sigma_{\text{exp}} \sqrt{2\pi}} \exp\left[-\frac{(\Gamma_{\text{exp}} - \Gamma(X_i))^2}{2 \sigma_{\text{exp}}^2}\right]
\]
- experimental distribution

\[
P(X_i) - \text{probability distribution of } X_i \text{ (prior)}
\]

- theoretical assumptions explicit and under control
- various assumptions testable

num.integration too demanding → Monte Carlo sampling

- 10000 samples per grid element, \(10^5-10^6\) total samples
- stability tested with smaller samples (1000 per grid element)
- in depth statistical stability test in preparation
C) Statistical analysis

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\[ P(X_i | \Gamma_{\text{exp}}) \] - probability density of \( X_i \) being true given \( \Gamma_{\text{exp}} \)

\[ P(\Gamma_{\text{exp}} | X_i) = \frac{1}{\sigma_{\text{exp}} \sqrt{2\pi}} \exp \left[ -\frac{(\Gamma_{\text{exp}} - \Gamma(X_i))^2}{\sigma_{\text{exp}}^2} \right] \] - experimental distribution

\[ P(X_i) \] - probability distribution of \( X_i \) (prior)

- theoretical assumptions explicit and under control
- various assumptions testable

num.integration too demanding \( \rightarrow \) Monte Carlo sampling

- 10000 samples per grid element, 10^5-10^6 total samples
- stability tested with smaller samples (1000 per grid element)
- in depth statistical stability test in preparation
D) Assumptions

$r=25$: motivated by lattice

$L_{1-3}$: mean and spread of a set of standard $\chi$PT fits:

\[
L_1^r(M_\rho) = (0.60 \pm 0.28) \cdot 10^{-3}
\]
\[
L_2^r(M_\rho) = (0.88 \pm 0.34) \cdot 10^{-3}
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L_3^r(M_\rho) = (-2.97 \pm 0.47) \cdot 10^{-3}
\]

weak dependence of the amplitude on $L_{1-3}$

$\Delta_k$: based on general arguments about the convergence of chiral series

\[
\Delta_G^{(4)} \approx \pm 0.3G,
\]
\[
\Delta_G^{(6)} \approx \pm 0.1G
\]

two implementations:

- normal distribution: $\mu=0$, $\sigma=0.1G$
- uniform distribution: $\Delta_G \in (-0.1G, 0.1G)$
- incompatible at $p \approx 0.68^{12} \approx 2.6\sigma$ level

$R$: (constraints on $X$ and $Z$)

$R = 37.8 \pm 3.3$

- beware: includes assumption that NNLO $S_\chi$PT converges well at a specific kinematic point

$X$: (constraints on $R$)

$X \in (0,1)$ or $X \in (0,0.9)$

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\( \eta \rightarrow 3\pi \): motivated by lattice \((\text{FLAG} \ 2011)\)

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E) Results - a first look

I. Constraints on $X$ and $Z$ - probability distribution ($Y = X/Z$)

Regions with $Y_{\text{max}} \leq 0.75$ and $Y_{\text{min}} \geq 2$:

- $p = 95.5\% = 2.0\sigma$
- $p = 99.9\% = 3.2\sigma$ vs. $p = 96.8\% = 2.1\sigma$
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\begin{align*}
X, Z \in (0, 1) & \\
X \in (0, 0.9), Z \in (0.5, 0.9) & \\
\end{align*}

not statistically significant

\begin{align*}
R > 44: & \quad p = 97.3\% = 2.2\sigma \\
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Our preliminary results have shown:

- the $\eta \to \pi^+\pi^-\pi^0$ decay width is sensitive to the value of the principal QCD order parameters, expressed in terms $X$ and $Z$
- a large portion of the parameter space can be excluded at $2.0\sigma$ C.L., given information about $R$
- $Y = X/Z \geq 1$ seems to be preferred
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Outlook - we plan include a wider range of experimental data, to extend the analysis to more parameters and to perform an in depth statistical stability test of the Monte Carlo sampling.
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