Transition form factors of $\pi^0$, $\eta$ and $\eta'$ mesons: What can be learned from anomaly sum rule?

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Outline

Motivation: problem with pion transition form factor

Anomaly Sum Rule

Isovector channel ($\pi^0$)

Corrections

Octet channel ($\eta, \eta'$)

Mixing

Summary
The measurements of the BABAR collaboration [Aubert et al. ’09] show a steady rise of $Q^2F_{\pi \gamma}$, surpassing the pQCD predicted asymptote $Q^2F_{\pi \gamma} \rightarrow \sqrt{2}f_\pi$, $f_\pi = 130.7$ MeV at $Q^2 \simeq 10$ GeV$^2$ and questioning the collinear factorization.

The BELLE data [Uehara et al. ’12] do not show such striking behavior: although $Q^2F_{\pi \gamma}$ reaches the pQCD asymptotic value, it does not manifest a further growth.
Axial anomaly

In QCD, for a given flavor $q$, the divergence of the axial current $J^{(q)}_{\mu 5} = \bar{q}\gamma_\mu\gamma_5 q$ acquires both electromagnetic and gluonic anomalous terms:

$$\partial_\mu J^{(q)}_{\mu 5} = m_q \bar{q}\gamma_5 q + \frac{e^2}{8\pi^2} e_q^2 N_c F \tilde{F} + \frac{\alpha_s}{4\pi} N_c G \tilde{G}, \quad (1)$$

An octet of axial currents

$$J^{(a)}_{\mu 5} = \sum_q \bar{q}\gamma_5 \gamma_\mu \frac{\lambda^a}{\sqrt{2}} q$$

Singlet axial current $J^{(0)}_{\mu 5} = \frac{1}{\sqrt{3}} (\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s)$:

$$\partial_\mu J^{(0)}_{\mu 5} = \frac{1}{\sqrt{3}} (m_u \bar{u}\gamma_5 u + m_d \bar{d}\gamma_5 d + m_s \bar{s}\gamma_5 s) + \frac{\alpha_{em}}{2\pi} C^{(0)} N_c F \tilde{F} + \frac{\sqrt{3}\alpha_s}{4\pi} N_c G \tilde{G}, \quad (2)$$
The diagonal components of the octet of axial currents

\[ J_{\mu 5}^{(3)} = \frac{1}{\sqrt{2}} (\bar{u} \gamma_\mu \gamma_5 u - \bar{d} \gamma_\mu \gamma_5 d), \]

\[ J_{\mu 5}^{(8)} = \frac{1}{\sqrt{6}} (\bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d - 2\bar{s} \gamma_\mu \gamma_5 s) \]

acquire an electromagnetic anomalous term only:

\[ \partial_\mu J_{\mu 5}^{(3)} = \frac{1}{\sqrt{2}} (m_u \bar{u} \gamma_5 u - m_d \bar{d} \gamma_5 d) + \frac{\alpha_{em}}{2\pi} C^{(3)} N_c F \tilde{F}, \] (3)

\[ \partial_\mu J_{\mu 5}^{(8)} = \frac{1}{\sqrt{6}} (m_u \bar{u} \gamma_5 u + m_d \bar{d} \gamma_5 d - 2m_s \bar{s} \gamma_5 s) + \frac{\alpha_{em}}{2\pi} C^{(8)} N_c F \tilde{F}. \] (4)

The electromagnetic charge factors \( C^{(a)} \) are

\[ C^{(3)} = \frac{1}{\sqrt{2}} (e_u^2 - e_d^2) = \frac{1}{3\sqrt{2}}, \]

\[ C^{(8)} = \frac{1}{\sqrt{6}} (e_u^2 + e_d^2 - 2e_s^2) = \frac{1}{3\sqrt{6}}, \]

\[ C^{(0)} = \frac{1}{\sqrt{3}} (e_u^2 + e_d^2 + e_s^2) = \frac{2}{3\sqrt{3}}. \] (5)
The matrix element for the transition of the axial current $J_{\alpha 5}$ with momentum $p = k + q$ into two real or virtual photons with momenta $k$ and $q$ is:

$$T_{\alpha \mu \nu}(k, q) = \int d^4x d^4y e^{(ikx + iqy)} \langle 0 | T \left\{ J_{\alpha 5}(0) J_\mu(x) J_\nu(y) \right\} | 0 \rangle; \quad (6)$$

Kinematics:

$$k^2 = 0, \quad Q^2 = -q^2$$
The VVA triangle graph amplitude can be presented as a tensor decomposition

\[ T_{\alpha\mu\nu}(k, q) = F_1 \varepsilon_{\alpha\mu\nu\rho} k^\rho + F_2 \varepsilon_{\alpha\mu\nu\rho} q^\rho + F_3 k_\nu \varepsilon_{\alpha\mu\rho\sigma} k^\rho q^\sigma + F_4 q_\nu \varepsilon_{\alpha\mu\rho\sigma} k^\rho q^\sigma + F_5 k_\mu \varepsilon_{\alpha\nu\rho\sigma} k^\rho q^\sigma + F_6 q_\mu \varepsilon_{\alpha\nu\rho\sigma} k^\rho q^\sigma, \]  

(7)

\[ F_j = F_j(p^2, k^2, q^2; m^2), \quad p = k + q. \]

Dispersive approach to axial anomaly leads to:

(real photons-[Dolgov, Zakharov'79], virtual photons- [Hořejší'85; Hořejší, Teryaev'94, Teryaev, Veretin'94]):

\[ \int_{4m^2}^{\infty} A_3(s, Q^2; m^2) ds = \frac{1}{2\pi} N_c C^{(a)}, \quad a = 3, 8; \]

(8)

\[ A_3 \equiv \frac{1}{2} \text{Im}(F_3 - F_6), \quad N_c = 3; \]

\[ C^{(3)} = \frac{1}{\sqrt{2}} (e_u^2 - e_d^2) = \frac{1}{3\sqrt{2}}, \]

\[ C^{(8)} = \frac{1}{\sqrt{6}} (e_u^2 + e_d^2 - 2e_s^2) = \frac{1}{3\sqrt{6}}, \]

(9)
\[
\int_{4m^2}^{\infty} A_3(s, Q^2; m^2)ds = \frac{1}{2\pi} N_c C^{(a)}
\]  

◮ Holds for any $Q^2$ and any $m^2$.

◮ It has neither $\alpha_s$ corrections (Adler-Bardeen theorem) nor nonperturbative corrections (t’Hooft’s consistency principle).

◮ Exact nonperturbative relation – powerful tool.
Transition form factors
Saturating the l.h.s. of the 3-point correlation function (6) with the resonances and singling out their contributions to ASR (10) we get the (infinite) sum of resonances with appropriate quantum numbers:

\[ \pi \sum f_M^a F_{M\gamma} = \int_{4m^2}^{\infty} A_3(s, Q^2; m^2) ds = \frac{1}{2\pi} N_c C^{(a)}, \]

where the coupling (decay) constants \( f_M^a \):

\[ \langle 0| J_{\alpha 5}^{(a)}(0)| M(p) \rangle = i p_\alpha f_M^a, \]

and form factors \( F_{M\gamma} \) of the transitions \( \gamma\gamma^* \rightarrow M \) are:

\[ \int d^4x e^{ikx} \langle M(p)| T \{ J_\mu(x) J_{\nu}(0) \} |0 \rangle = \epsilon_{\mu\nu\rho\sigma} k^\rho q^\sigma F_{M\gamma} \]

- Sum of finite number of resonances decreasing \( F_{M\gamma}^{\text{asymp}}(Q^2) \propto \frac{f_M}{Q^2} \)
- Infinite number of states are needed to saturate ASR (collective effect). [Y.K., A.Oganesian, O.Teryaev’10]
Isovector channel: \( \pi^0 \)

\[
J_{\mu 5}^{(3)} = \frac{1}{\sqrt{2}} (\bar{u} \gamma_\mu \gamma_5 u - \bar{d} \gamma_\mu \gamma_5 d), \quad C^{(3)} = \frac{1}{\sqrt{2}} (e_u^2 - e_d^2) = \frac{1}{3\sqrt{2}}.
\]

For practical purposes let’s use QHD and describe the higher resonances by continuum.

\[\pi^0 + \text{higher contributions ("continuum")}:\]

\[
\pi f_\pi F_{\pi \gamma}(Q^2) + \int_{s_0}^{\infty} A_3(s, Q^2; m^2) = \frac{1}{2\pi} N_c C^{(3)}. \quad (14)
\]

The spectral density \( A_3(s, Q^2; m^2) \) can be calculated from VVA triangle diagram:

\[
A_3(s, Q^2; m^2) = \frac{1}{2\sqrt{2}\pi} \frac{1}{(Q^2 + s)^2} \left( Q^2 R + 2m^2 \ln \frac{1 + R}{1 - R} \right), \quad (15)
\]

where \( R(s, m) = \sqrt{1 - \frac{4m^2}{s}} \), \( m \) is a mass of quark. Then the pion TFF:

\[
F_{\pi \gamma}(Q^2) = \frac{1}{2\sqrt{2}\pi^2 f_\pi} \frac{s_0}{s_0 + Q^2} \left( R_0 - \frac{2m^2}{s_0} \ln \frac{1 + R_0}{1 - R_0} \right), \quad (16)
\]

\( R_0 = R(s_0, m). \)
\[ m = 0: \]
\[ F_{\pi\gamma}(Q^2) = \frac{1}{2\sqrt{2}\pi^2 f_{\pi}} \frac{s_0}{s_0 + Q^2} \]  

(17)

[Cf. A. Aleksejevs’ talk].

Considering the limit \( Q^2 \to \infty \) and relying the QCD factorization prediction for \( Q^2 F_{\pi\gamma} = \sqrt{2} f_{\pi} (\phi^{as}(x) = 6x(1-x)) \)

\[ s_0 = 4\pi^2 f_{\pi}^2 = 0.67 \text{GeV}^2 \]

– fits perfectly the value extracted from SVZ (two-point) QCD sum rules \( s_0 = 0.7 \text{GeV}^2 \) [Shifman, Vainshtein, Zakharov’79].

– gives a proof for BL interpolation formula [Brodsky, Lepage’81]:

\[ F_{\pi\gamma}^{\text{BL}}(Q^2) = \frac{1}{2\sqrt{2}\pi^2 f_{\pi}} \frac{1}{1 + Q^2 / (4\pi^2 f_{\pi}^2)}. \]  

(18)

Derived first from QHD [Radyushkin'96], now we related it to anomaly at all \( Q^2 \).
The full integral is exact

\[ \frac{1}{2\sqrt{2\pi}} = \int_0^\infty A_3(s, Q^2) ds = I_\pi + I_{cont} \]

The continuum contribution \( I_{cont} = \int_{s_0}^\infty A_3(s, Q^2) ds \) may have perturbative as well as power corrections originating from corrections to \( A_3 \).

\( \delta I_\pi = -\delta I_{cont} \): small relative correction to continuum – due to exactness of ASR – must be compensated by large relative correction to the pion contribution!
Possible corrections to $A_3$

- Perturbative two-loop corrections to spectral density $A_3$ are zero [Jegerlehner&Tarasov’06]

- Nonperturbative corrections to $A_3$ are possible: vacuum condensates, instantons, short strings [Chetyrkin,Zakharov’98].

- General requirements for the correction $\delta l = \int_{s_0}^{\infty} \delta A_3(s, Q^2) ds$:
  \[ \delta l = 0 \]
  - at $s_0 \to \infty$ (the continuum contribution vanishes),
  - at $s_0 \to 0$ (the full integral has no corrections),
  - at $Q^2 \to \infty$ (the perturbative theory works at large $Q^2$ ),
  - at $Q^2 \to 0$ (anomaly perfectly describes pion decay width).

\[ \delta l = \frac{1}{2\sqrt{2\pi}} \frac{s_0 Q^2}{(s_0 + Q^2)^2} \left( \ln \frac{Q^2}{s_0} + \sigma \right), \]  \hspace{1cm} (19)

\[ \delta F_{\pi\gamma} = \frac{1}{\pi f_\pi} \delta l_{\pi} = \frac{1}{2\sqrt{2\pi^2 f_\pi}} \frac{s_0 Q^2}{(s_0 + Q^2)^2} \left( \ln \frac{Q^2}{s_0} + \sigma \right). \]  \hspace{1cm} (20)
CELLO + CLEO + BABAR: $\lambda = 0.14$, $\sigma = -2.36$, $\chi^2 / n.d.f. = 0.94$
\[ \chi^2_{n,d.f.} \]

\begin{tabular}{|c|c|c|c|c|}
\hline
 & \( \chi^2_{n,d.f.} \) & \( \delta l \neq 0 \) & \( \chi^2_{n,d.f.} \) & \( \lambda \) & \( \sigma \) \\
\hline
CELSO+CLEO+BABAR+Belle & 1.86 & 0.91 & 0.12 & -2.50 \\
CELSO+CLEO+Belle & 1.01 & 0.46 & 0.07 & -3.03 \\
CELSO+CLEO+BABAR & 2.29 & 0.94 & 0.14 & -2.36 \\
BABAR & 3.61 & 0.99 & 0.20 & -2.39 \\
Belle & 0.80 & 0.40 & 0.14 & -2.86 \\
\hline
\end{tabular}

> Although the BELLE data themselves may be described without the correction, but they do not also exclude its possibility. Unless the BABAR data will be disproved, the need for correction remains.
Octet channel \((\eta, \eta')\)

\[
J^{(8)}_{\alpha 5} = \frac{1}{\sqrt{6}}(\bar{u}\gamma_\alpha\gamma_5 u + \bar{d}\gamma_\alpha\gamma_5 d - 2\bar{s}\gamma_\alpha\gamma_5 s),
\]

\[
\int_{4m^2}^{\infty} A_3(s, Q^2; m^2)ds = \frac{1}{2\pi} N_c C^{(8)},
\]

\[
C^{(8)} \equiv \frac{1}{\sqrt{6}}(e_u^2 + e_d^2 - 2e_s^2) = \frac{1}{3\sqrt{6}}
\]

ASR in the octet channel:

\[
f^8_\eta F_{\eta\gamma}(Q^2) + f^8_{\eta'} F_{\eta'\gamma}(Q^2) = \frac{1}{2\sqrt{6}\pi^2} \frac{s_0}{s_0 + Q^2}
\]

- Significant mixing.
- \(\eta'\) decays into two real photons, so it should be taken into account explicitly along with \(\eta\) meson.
Large $Q^2$

[Anisovich, Melikhov, Nikonov’97; Feldmann, Kroll’98]

$$Q^2 F_{\eta\gamma}^{as} = 2 \left( C^{(8)} f_{\eta}^8 + C^{(0)} f_{\eta}^0 \right) \int_0^1 \frac{\phi^{as}(x)}{x} dx,$$  \hspace{1cm} (23)

$$Q^2 F_{\eta'\gamma}^{as} = 2 \left( C^{(8)} f_{\eta'}^8 + C^{(0)} f_{\eta'}^0 \right) \int_0^1 \frac{\phi^{as}(x)}{x} dx,$$  \hspace{1cm} (24)

$$\phi^{as}(x) = 6x(1 - x).$$  \hspace{1cm} Then ASR at $Q^2 \rightarrow \infty$:

$$4\pi^2 \left( (f_{\eta}^8)^2 + (f_{\eta'}^8)^2 + 2\sqrt{2}[f_{\eta}^8 f_{\eta'}^0 + f_{\eta}^0 f_{\eta'}^0] \right) = s_0.$$  \hspace{1cm} (25)
$Q^2 = 0$

ASR takes the form:

$$f_\eta^8 F_{\eta\gamma}(0) + f_{\eta'}^8 F_{\eta'\gamma}(0) = \frac{1}{2\sqrt{6}\pi^2},$$

(26)

where

$$F_{M\gamma}(0) = \sqrt{\frac{4\Gamma_{M\rightarrow\gamma\gamma}}{\pi\alpha^2 m^3_M}}.$$
Mixing

PCAC relation for $\pi^0$

$$\partial_\mu J^{(3)}_{\mu_5} = f^{(3)}_\pi m^2_\pi \phi_\pi.$$ (27)

Generalization to the mixing system $\eta - \eta' - ...$

[cf. Ioffe '79]

$$\partial_\mu J_{\mu_5} = FM\Phi,$$ (28)

where:

$$J_{\mu_5} \equiv \begin{pmatrix} J^\alpha_{\mu_5} \\ J^\beta_{\mu_5} \end{pmatrix}, \quad F \equiv \begin{pmatrix} f^\alpha_\eta & f^\alpha_\eta' & f^\alpha_G & \cdots \\ f^\beta_\eta & f^\beta_\eta' & f^\beta_G & \cdots \end{pmatrix}, \quad \Phi \equiv \begin{pmatrix} \phi_\eta \\ \phi_\eta' \\ \phi_G \\ \cdots \end{pmatrix},$$

$$M \equiv \text{diag}(m^2_\eta, m^2_\eta', m^2_G, \cdots).$$ (29)

$$\langle 0|J^{(a)}_{\alpha_5}(0)|M(p)\rangle = ip_\alpha f^a_M.$$
Mixing

Octet-singlet basis (of currents):

\[ J^{(8)}_{\mu 5} = \frac{1}{\sqrt{6}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d - 2\bar{s}\gamma_\mu\gamma_5 s), \quad J^{(0)}_{\mu 5} = \frac{1}{\sqrt{3}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s). \]  

(30)

For quark-flavor basis one explores the definitions of axial currents with decoupled light and strange quark composition:

\[ J^q_{\mu 5} = \frac{1}{\sqrt{2}}(\bar{u}\gamma_\alpha\gamma_5 u + \bar{d}\gamma_\alpha\gamma_5 d), \quad J^s_{\mu 5} = \bar{s}\gamma_\alpha\gamma_5 s, \]

(31)

\[
\begin{pmatrix}
  J^8_{\mu 5} \\
  J^0_{\mu 5}
\end{pmatrix} = \mathbf{V}(\alpha) \begin{pmatrix}
  J^q_{\mu 5} \\
  J^s_{\mu 5}
\end{pmatrix}, \quad \mathbf{V}(\alpha) = \begin{pmatrix}
  \cos \alpha & -\sin \alpha \\
  \sin \alpha & \cos \alpha
\end{pmatrix},
\]

(32)

where \( \tan \alpha = \sqrt{2} \).
\[ \tilde{\Phi} = U\Phi, \quad \tilde{\Phi} \equiv \begin{pmatrix} \tilde{\phi}_\alpha \\ \tilde{\phi}_\beta \\ \tilde{\phi}_G \\ \vdots \end{pmatrix}. \tag{33} \]

In terms of these fields, Eq. (28) can be rewritten as

\[ \partial_\mu J_{\mu 5} = \tilde{F}\tilde{M}\tilde{\Phi}, \tag{34} \]

where \( \tilde{F} = FU, \quad \tilde{M} = U^T MU. \)

In our notations the octet-singlet (quark-flavor) mixing scheme implies that the matrix \( \tilde{F} \) has a (rectangular) diagonal form in the respective octet-singlet (quark-flavor) basis,

\[ \tilde{F} = \begin{pmatrix} f_\alpha & 0 & 0 & \ldots \\ 0 & f_\beta & 0 & \ldots \end{pmatrix}. \tag{35} \]

\( FU \) has a (rectangular) diagonal form (35) immediately follows that \( FF^T \)
is a diagonal matrix.
So, imposing the mixing scheme is equivalent to imposing the constraint for the decay constants: [YK, Oganesian, Teryaev'2012]

\[ f^\alpha_{\eta} f^{\beta}_{\eta} + f^\alpha_{\eta'} f^{\beta}_{\eta'} + f^\alpha_{G} f^{\beta}_{G} + ... = 0. \]  \hspace{1cm} (36)

- **Octet-singlet (SU(3)) mixing scheme:** \( f^8_{\eta} f^0_{\eta} + f^8_{\eta'} f^0_{\eta'} = 0. \)

\[
F = \begin{pmatrix}
    f_8 \cos \theta & f_8 \sin \theta \\
    -f_0 \sin \theta & f_0 \cos \theta
\end{pmatrix}.
\]  \hspace{1cm} (37)

- **Quark-flavour mixing scheme:**

\[ f^{q}_{\eta} f^{s}_{\eta} + f^{q}_{\eta'} f^{s}_{\eta'} = 0. \]

[Feldmann,Kroll,Stech'97]

\[
F_{qs} = \begin{pmatrix}
    f_q \cos \phi & f_q \sin \phi \\
    -f_s \sin \phi & f_s \cos \phi
\end{pmatrix}.
\]  \hspace{1cm} (38)
The interplay of two- and three-point correlators was investigated for the case of isovector current and pion state in [Radyushkin'95] and the duality interval was expressed in terms of pion decay constant $f_\pi = 0.13$ GeV:

$$s_3^\pi = 4\pi^2 f_\pi^2.$$  
(39)

"$\eta + \text{continuum}$":

$$s_8^\eta = 4\pi^2 (f_\eta^8)^2.$$  
(40)

"$\eta + \eta' + \text{continuum}$":

$$s_8^{\eta + \eta'} = 4\pi^2 ((f_\eta^8)^2 + (f_{\eta'}^8)^2).$$  
(41)

At the same time from 3-point correlator:

$$s_8^{asr} = 4\pi^2 ((f_\eta^8)^2 + (f_{\eta'}^8)^2 + 2\sqrt{2}[f_\eta^8 f_0^\eta + f_{\eta'}^8 f_0^{\eta'}]).$$  
(42)
Additional constraint - $R_{J/\psi}$.
The radiative decays $J/\psi \to \eta(\eta')\gamma$ are dominated by non-perturbative gluonic matrix elements, and the ratio of the decay rates

$$R_{J/\psi} = (\Gamma(J/\psi \to \eta'\gamma)/(\Gamma(J/\psi \to \eta\gamma))$$

can be expressed as follows [Novikov'79]:

$$R_{J/\psi} = \left| \left\langle 0 \left| \hat{G} \tilde{G} \right| \eta' \right\rangle / \left\langle 0 \left| \hat{G} \tilde{G} \right| \eta \right\rangle \right|^2 \left( \frac{p_{\eta'}}{p_\eta} \right)^3,$$

where $p_{\eta(\eta')} = M_{J/\psi}(1 - m_{\eta(\eta')}/M_{J/\psi}^2)/2$.

$$\partial_\mu J_{\mu 5}^8 = \frac{1}{\sqrt{6}}(m_u \bar{u}_5 u + m_d \bar{d}_5 d - 2m_s \bar{s}_5 s),$$

$$\partial_\mu J_{\mu 5}^0 = \frac{1}{\sqrt{3}}(m_u \bar{u}_5 u + m_d \bar{d}_5 d + m_s \bar{s}_5 s) + \frac{1}{2\sqrt{3}} \frac{3\alpha_s}{4\pi} G \tilde{G}.$$ (45)

$$R_{J/\psi} = \left( \frac{f_{\eta'}^8 + \sqrt{2}f_{\eta'}^0}{f_\eta^8 + \sqrt{2}f_\eta^0} \right)^2 \left( \frac{m_{\eta'}}{m_\eta} \right)^4 \left( \frac{p_{\eta'}}{p_\eta} \right)^3.$$ (46)

From experiment this ratio is: $R_{J/\psi} = 4.67 \pm 0.15$ [PDG 2012].
$$f_\eta^8 F_{\eta\gamma}(Q^2) + f_{\eta'}^8 F_{\eta'\gamma}(Q^2) = \frac{1}{2\sqrt{6}\pi^2} \frac{s_0}{s_0 + Q^2},$$

(47)

$$4\pi^2((f_\eta^8)^2 + (f_{\eta'}^8)^2 + 2\sqrt{2}[f_\eta^8 f_0^0 + f_{\eta'}^8 f_{\eta'}^0]) = s_0,$$

(48)

$$f_\eta^8 F_{\eta\gamma}(0) + f_{\eta'}^8 F_{\eta'\gamma}(0) = \frac{1}{2\sqrt{6}\pi^2}, F_{M\gamma}(0) = \sqrt{\frac{4\Gamma_{M\rightarrow\gamma\gamma}}{\pi\alpha^2 m_M^3}}$$

(49)

$$R_{J/\Psi} = \left(\frac{f_{\eta'}^8 + \sqrt{2}f_{\eta'}^0}{f_\eta^8 + \sqrt{2}f_\eta^0}\right)^2 \left(\frac{m_{\eta'}}{m_\eta}\right)^4 \left(\frac{p_{\eta'}}{p_\eta}\right)^3.$$  

(50)
TFFs of $\eta$ and $\eta'$: experiment

Figure: Experimental data on transition form factors: $\eta$ (CLEO-green, BABAR-blue), $\eta'$ (CLEO-pink, BABAR-red)
$f_8/f_\pi = 0.88 \pm 0.04, \quad f_0/f_\pi = 0.81 \pm 0.07, \quad \theta = -14.2^\circ \pm 0.7^\circ$
Quark-flavour mixing scheme: parameters

\[ f_q/f_\pi = 1.20 \pm 0.15, \quad f_s/f_\pi = 1.65 \pm 0.25, \quad \phi = 38.1^\circ \pm 0.5^\circ \]
Mixing-scheme-independent determination

\[ F = \begin{pmatrix} f_\eta^8 & f_\eta' \\ f_\eta^0 & f_\eta'^0 \end{pmatrix} = \begin{pmatrix} 1.11 & -0.42 \\ 0.16 & 1.04 \end{pmatrix} f_\pi \]  

(51)
The similar correction in the octet channel leads to the ASR:

\[
f^8_\eta F_{\eta\gamma}(Q^2) + f^8_{\eta'} F'_{\eta'\gamma}(Q^2) = \frac{1}{2\sqrt{6}\pi^2} \frac{s_8}{s_8 + Q^2} \left[ 1 + \frac{\lambda Q^2}{s_8 + Q^2} (\ln \frac{Q^2}{s_8} + \sigma) \right].
\]
Summary

- The manifestation of Axial Anomaly – the Anomaly Sum Rule is an exact NPQCD tool applicable even beyond QCD factorization.

- Anomaly sum rule allows to derive an expression for the pion transition form factor at arbitrary $Q^2$, giving the proof for the Brodsky-Lepage interpolation formula.

- In order to describe the BABAR data on pion TFF, ASR requires a new nonperturbative correction to the spectral density. At the same time, the BELLE data can be described well without such a correction. The same log-like correction can be accommodated in the octet channel, although there is no strong indication of it (more precise experimental data on $\eta, \eta'$ TFFs are needed).

- ASR allows to obtain constraints for the $\eta – \eta'$ mixing parameters in different mixing schemes as well as independently of the schemes. The future improvement of experimental data on transition form factors of $\eta, \eta'$ mesons and ratio $R_{J/\psi}$ can determine which mixing scheme is valid.
Thank you for your attention!
Bose symmetry $T_{\alpha \mu \nu}(k, q) = T_{\alpha \nu \mu}(q, k)$ implies:

$$F_1(k, q) = -F_2(q, k),$$
$$F_3(k, q) = -F_6(q, k),$$
$$F_4(k, q) = -F_5(q, k).$$

One can show also that

$$F_6(k, q) = -F_3(k, q).$$

Vector Ward identities

$$k^\mu T_{\alpha \mu \nu} = 0, \quad p^\nu T_{\alpha \mu \nu} = 0$$

In terms of formfactors, the identities (53) read

$$F_1 = k \cdot q F_3 + q^2 F_4$$
$$F_2 = k^2 F_5 + k \cdot q F_6$$

Anomalous axial-vector Ward identity for the amplitude (7) is [Adler’69]:

$$q^\alpha T_{\alpha \mu \nu}(k, q) = 2mT_{\mu \nu}(k, q) + \frac{1}{2\pi^2} \epsilon_{\mu \nu \rho \sigma} k^\rho q^\sigma$$
\[ T_{\mu\nu}(k, q) = G \varepsilon_{\mu\nu\rho\sigma} k^\rho q^\sigma \quad (56) \]

where \( G \) is the relevant form factor. In terms of form factors, eq.(55) reads

\[ F_2 - F_1 = 2mG + \frac{1}{2\pi^2} \quad (57) \]

For the form factors \( F_3, F_4 \) and \( G \) one may write unsubtracted dispersion relations

\[ F_j(p^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{A_j(t)}{t - p^2} dt, \quad j = 3, 4 \quad (58) \]

\[ G(p^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{B(t)}{t - p^2} dt \]
From (34) and (54) it is easy to see that for the considered kinematical configuration one has

$$F_2 - F_1 = (q^2 - p^2)F_3 - p^2 F_4$$  \hspace{1cm} (59)$$

Using now (58) and taking into account that the imaginary parts of the relevant formfactors satisfy non-anomalous Ward identities, in particular

$$(q^2 - t)A_3(t) - q^2 A_4(t) = 2mB(t)$$  \hspace{1cm} (60)$$

one gets finally

$$F_2 - F_1 - 2mG = \frac{1}{\pi} \int_{4m^2}^{\infty} A_3(t)dt$$  \hspace{1cm} (61)$$

Comparing eq.(61) with (57) one may thus observe that the occurrence of the axial anomaly is equivalent to a "sum rule"

$$\int_{4m^2}^{\infty} A_3(t, q^2; m^2)dt = \frac{1}{2\pi}$$  \hspace{1cm} (62)$$

(which must hold for an arbitrary $m$ and for any $p^2$ in the considered region).