Scalar and tensor $\pi\pi$ resonances in dispersive analysis

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Stary Smokovec, Slovakia
Puzzling $S0$ wave $\pi\pi$ cross section

\[ \sigma_{\pi\pi} \text{ (mb)} \]

\[ f_0(600)/\sigma \quad f_0(1370) \]

\[ f_0(980) \quad f_0(1500) \]

\[ m_{\pi\pi} \text{ (MeV)} \]
rich but difficult life of the $\sigma$ meson

- until 1976 called $\epsilon$ or $\sigma$,
- excluded from Particle Data Tables from 1978 to 1992 and replaced by correlated two pions,
- since 1994: $f_0(400 - 1200)$,
- in years 2002-2010: $f_0(600)$,
- now (since 2012): $f_0(500)$
2005: can we do something?
... so let’s start with ..... Roy eqs
Dispersion relations with imposed crossing symmetry condition for $\pi\pi$ interactions theory $\leftrightarrow$ experiment

Crossing symmetry:

$$\mathbf{T}_s(s, t) = \hat{C}_{st} \mathbf{T}_t(t, s)$$

$\mathbf{T}(s, t) +$ crossing symmetry $\rightarrow$ dispersion relations for $4m^2_\pi < s < \sim (1150 \text{ MeV})^2$
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Once subtracted DR:

$$\text{Re } \vec{F}(s, t) = \text{Re } \vec{F}(s_0, t) + \frac{s - s_0}{\pi}$$

$$\times \left[ \int_{-\infty}^{\infty} ds' \frac{\text{Im } \vec{F}(s', t)}{(s' - s_0)(s' - s)} \right]$$

$$+ \int_{-t}^{\infty} ds' \frac{\text{Im } \vec{F}(s', t)}{(s' - s_0)(s' - s)}$$
Dispersion relations with imposed crossing symmetry condition for $\pi\pi$ interactions theory $\longleftrightarrow$ experiment

Crossing symmetry:

$\mathbf{T}(s, t) + \text{crossing symmetry} \rightarrow \text{dispersion relations for } 4m_\pi^2 < s < \sim (1150 \text{ MeV})^2$

Once subtracted dispersion relations ("GKPY" for the $S$ and $P$ waves):

$$\text{Re} \ t^{(OUT)}_\ell(s) = \sum_{l'=0}^{2} C^{l'l'}_{\text{st}} a'^{l'}_0 + \sum_{l'=0}^{2} \sum_{l''=0}^{4} \int_{4m_\pi^2}^{\infty} ds' K^{l'l'}_{\ell\ell'}(s, s') \text{Im} \ t^{(IN)}_{\ell'}(s')$$

$a'^{l'}_0$ - subtraction constant $= \mathbf{T}_s(s = 4m_\pi^2, t = 0)$ - scattering lengths from only $S$ wave due to $\text{Re} \ t^l_k(k) = k^{2\ell}(a^l_k + b^l_k k^2 + O(k^4))$
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\]

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due to $\operatorname{Re} t^{l}_\ell(k) = k^{2l}(a^l_\ell + b^l_\ell k^2 + O(k^4))$

\[
\operatorname{Re} t^{(\text{OUT})}_\ell(s) - \operatorname{Re} t^{(\text{IN})}_\ell(s) \rightarrow 0
\]
Experimental data for the \(\pi \pi\) in the \(S0\) wave (JI)

\[\text{Im } t_{\ell'}^{(IN)}(s') \sim \eta(s') \cos(2\delta(s'))\]
Experimental data for the $\pi\pi$ in the $S0$ wave (JI)

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$$\text{Im } t_{\ell'}^{(\text{IN})}(s') \sim \eta(s') \cos(2\delta(s'))$$
Experimental data for the $\pi\pi$ in the $S0$ wave (JI)

$$\Im m_{t_s^{(IN)}}(s') \sim \eta(s')\cos(2\delta(s'))$$
"Precise determination of the f0(500) and f0(980) pole parameters from a dispersive data analysis","R. Garcia-Martin, R. Kamiński, J.R. Pelaez, J. Ruiz de Elvira, Phys. Rev. Lett. 107 (2011) 072001

precision of the Roy and GKPY equations
precise determination of $f_0(500)$ ($\sigma$) meson and threshold parameters

$f_0(500)$ ($\sigma$)

- **PDG 2010:**
  - $M = 400 - 1200$ MeV
  - $\Gamma = 2 \times (250 - 500)$ MeV

- **PDG 2012:**
  - $M = 400 - 550$ MeV
  - $\Gamma = 2 \times (200 - 350)$ MeV

- **GKPY:**
  - $E_\sigma = 457 \pm 14 - i279^{+11}_{-7}$ MeV

threshold parameters, e.g. $a_0^0$:

- **ChPT + Roy eqs (Bern group):**
  - $0.220 \pm 0.005 \, m_\pi^{-1}$

- **GKPY:**
  - $0.220 \pm 0.008 \, m_\pi^{-1}$
**precise determination of \( f_0(500) (\sigma) \) meson and threshold parameters**

\( f_0(500) (\sigma) \)

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- ChPT + Roy eqs (Bern group):
  \[ 0.220 \pm 0.005 \text{ } m_\pi^{-1} \]

- **GKPY:**
  \[ 0.220 \pm 0.008 \text{ } m_\pi^{-1} \]
Before 2012

\( f_0(600) \)

or \( \sigma \)

A REVIEW GOES HERE – Check our WWW

\( f_0(600) \) T-MATRIX POLE \( \sqrt{s} \)

Note that \( \Gamma \approx 2 \text{ Im}(\sqrt{s_{\text{pole}}}) \).

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<td>(533 ± 25)–( i(247 ± 25) )</td>
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<td>(532 –i272)</td>
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<td>(470 ± 30)–( i(295 ± 20) )</td>
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Since year 2012

\( f_0(500) \) or \( \sigma \)

was \( f_0(600) \)

A REVIEW GOES HERE – Check our WWW

\( f_0(500) \) T-MATRIX POLE \( \sqrt{s} \)

Note that \( \Gamma \approx 2 \text{ Im}(\sqrt{s_{\text{pole}}}) \).

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<td>(445 ± 25)–( i(278 ± 22) )</td>
<td>2,3</td>
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<td>(457 +14−13)–( i(279 ± 11) )</td>
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<td>(442 +5−8)–( i(274 ± 6) )</td>
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<td>(452 ± 13)–( i(259 ± 16) )</td>
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<td>(448 ± 43)–( i(266 ± 43) )</td>
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<td>(466 ± 18)–( i(223 ± 28) )</td>
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<td>(472 ± 30)–( i(271 ± 30) )</td>
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<td>(484 ± 17)–( i(255 ± 10) )</td>
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precise determination of couplings to the $\pi\pi$ channel

\[ g^2 = -16\pi \lim_{s \to s_{\text{pole}}} (s - s_{\text{pole}}) t_\ell(s)(2\ell + 1)/(2p)^{2\ell} \]

where $p^2 = s/4 - m^2_\pi$.

| Resonance | $\sqrt{s_{\text{pole}}}$ (MeV) | $|g|$ |
|-----------|---------------------------------|------|
| $f_0(500)^{\text{GKPY}}$ | (457$^{+14}_{-13}$) $-$ $i$(279$^{+11}_{-7}$) | 3.59$^{+0.11}_{-0.13}$ GeV |
| $f_0(500)^{\text{Roy}}$ | (445 $\pm$ 25) $-$ $i$(278$^{+22}_{-18}$) | 3.4 $\pm$ 0.5 GeV |
| $f_0(980)^{\text{GKPY}}$ | (996 $\pm$ 7) $-$ $i$(25$^{+10}_{-6}$) | 2.3 $\pm$ 0.2 GeV |
| $f_0(980)^{\text{Roy}}$ | (1003$^{+5}_{-27}$) $-$ $i$(21$^{+10}_{-8}$) | 2.5$^{+0.2}_{-0.6}$ GeV |
| $\rho(770)^{\text{GKPY}}$ | (763.7$^{+1.7}_{-1.5}$) $-$ $i$(73.2$^{+1.0}_{-1.1}$) | 6.01$^{+0.04}_{-0.07}$ |
| $\rho(770)^{\text{Roy}}$ | (761$^{+4}_{-3}$) $-$ $i$(71.7$^{+1.9}_{-2.3}$) | 5.95$^{+0.12}_{-0.08}$ |
single pole - closest to physical region one

Unitarity: $S(k) = S^*(−k^*)$

$S = \frac{−k−p}{k−p} \times \frac{−k−p′}{k−p′}$

Pole $p'$: needed for unitarity but gives also a small contribution to the phase shifts.
how important is pole $\rho'$?

exercise with $\rho(770)$ (P-wave)

PDG Tables’2012:
$M = 771.1 \pm 0.9$ MeV,
$\Gamma = 149.2 \pm 0.7$ MeV

It corresponds to $\delta = 90^\circ$ and includes influence of the symmetric pole $\rho'$.

Analytical continuation of the amplitude to the complex $s$ plane gives
$M = 766.1$ MeV, $\Gamma = 149.2$ MeV
for the single $\rho$ pole.

Some analyses find difference $\sim 10$ MeV for mass ($Res_{\rho\rho}$) of the $\rho$. 
... left cut is enough, we do not need GKPY ...

Left hand cut in parameterizations of amplitudes:

- additional factor $e^{i\alpha}$ in the full $S = e^{2i\delta}$ matrix element,
- It has, however, nothing to do with crossing symmetry!
  - It does not provide any type of relationship $A(s, t) = C_{st}A(t, s)$,
  - Moreover, subtracting constant is not specified so the output amplitude can be arbitrarily scaled!
- it makes amplitude only more realistic
what forces GKPY eqs to pull up-left the sigma pole?

Two things: trigonometry and crossing symmetry algebra lead to narrower and lighter $\sigma$. 
what forces GKPY eqs to pull up-left the sigma pole?

Two things: trigonometry and crossing symmetry algebra lead to narrower and lighter $\sigma$.

Nothing more and nothing instead of it is needed.
what forces GKPY eqs to pull up-left the sigma pole?

\[
\text{Re } t^{(\text{OUT})}_\ell(s) = \sum_{I' = 0}^{2} C^{ll'} t^{(\text{IN})}_{0}(4m^2) + \sum_{I' = 0}^{2} \sum_{\ell' = 0}^{4} \int_{0}^{\infty} ds' K^{ll'}_{\ell\ell'}(s, s') \text{Im } t^{(\text{IN})}_{\ell'}(s')
\]

\[
\text{Re } t^{0(\text{OUT})}_{0}(s) = \text{Re } t^{0(\text{IN})}_{0}(s)
\]
what forces GKPY eqs to pull up-left the sigma pole?

\[
\text{Re } t_{l}^{(\text{OUT})}(s) = \sum_{l'=0}^{2} C^{ll'} t_{0}^{(\text{IN})}(4m_{\pi}^2) + \sum_{l'=0}^{2} \sum_{\ell'=0}^{4} \int_{4m_{\pi}^2}^{\infty} \text{d}s' K_{ll'}^{ll'}(s, s') \text{Im } t_{l'}^{(\text{IN})}(s')
\]

\[
\text{Re } t_{0}^{(\text{OUT})}(s) = \text{Re } t_{0}^{(\text{IN})}(s)
\]
What does lead to such shape of the $KT_{00}^{00}$?

The shape is given by coefficients in the crossing symmetry matrix $C_{st}$ and integrated amplitudes. Is it produced by the integration along the left or right cut?
Independent mathematical confirmation


In the recently elaborated fully solvable mathematical problem, to be concerned of a finding of an explicit form of the pion scalar form factor, the inaccurate experimental information on the S-wave iso-scalar $\pi\pi$-scattering phase shift in the elastic region is replaced by the data with theoretical errors to be generated by the Garcia-Martin-Kaminski-Pelaez-Yndurain Roy-like equations and as a result the correct values $m_\sigma = (472 \pm 10)$ MeV and $\Gamma_\sigma = (524 \pm 22)$ MeV of the scalar meson $f_0(500)$ parameters are determined.
Fig. 1. (Left) $\alpha_\rho(s)$ and $\alpha_\sigma(s)$ Regge trajectories, from our constrained Regge-pole amplitudes. (Right) $\alpha_\sigma(s)$ and $\alpha_\rho(s)$ in the complex plane. At low and intermediate energies (thick continuous lines), the trajectory of the $\sigma$ is similar to those of Yukawa potentials $V(r) = -G \alpha \exp(-r/a)/r$ [8] (thin dashed lines). Beyond 2 GeV$^2$ we plot our results as thick discontinuous lines because they should be considered just as extrapolations.

Furthermore, in Fig. 1 we show the striking similarities between the $f_0(500)$ trajectory and those of Yukawa potentials in non-relativistic scattering [8]. From the Yukawa $G=2$ curve in that plot, which lies closest to our result for the $f_0(500)$, we can estimate $a \simeq 0.5$ GeV$^{-1}$, following [8]. This could be compared, for instance, to the S-wave $\pi\pi$ scattering length $\simeq 1.6$ GeV$^{-1}$. Thus it seems that the range of a Yukawa potential that would mimic our low energy results is comparable but smaller than the $\pi\pi$ scattering length in the scalar isoscalar channel. Of course, our results are most reliable at low energies (thick continuous line) and the extrapolation should be interpreted cautiously. Nevertheless, our results suggest that the $f_0(500)$ looks more like a low-energy resonance of a short range potential, e.g. between pions, than a bound state of a confining force between a quark and an antiquark.

In summary, our formalism and the results for the $f_0(500)$ explains why the lightest scalar meson has to be excluded from the ordinary linear Regge fits of ordinary mesons.
Results for the $I^G J^{PC} : 0^{++} 2^{++}$ and $1^{+} 3^{--}$ mesons

**D-wave:**

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<th>$\chi^2$</th>
<th>$\chi^2_{\text{Data}}$</th>
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<td>409.8</td>
<td>270.8</td>
<td>139.1</td>
<td>1.5</td>
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$f_2(1270), f_2(1430), f_2(1525), f_2(1600), f_2(1730), f_2(1810), f_2(1960), f_2(2000), f_2(2020), f_2(2240), f_2(2410)$

**F-wave:**

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<td>753.6</td>
<td>489.3</td>
<td>129.4</td>
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$f_3(1690), f_3(1950)$
Conclusions

- **unitarity, analyticity** and **crossing symmetry** applied to $\pi\pi$ amplitudes lead to very precise and not biased amplitudes,
- these very well agree with experimental data below 2000 MeV,
- Regge trajectories enhance this region to 20 GeV,
- all ($S...F$) partial waves depend and agree with each other,
- finally we have a very precise and uncompromising tool to test various model $\pi\pi$ amplitudes,
- finally one can study $f_0$... resonances,
- search for exotic particles at JLab (GlueX and CLAS12) can safely begin
Conclusions

- The $\sigma$ meson is once again alive and is doing well!

- For sure $\sigma$ is not pure $q\bar{q}$ meson but perhaps:
  - Mixture of the $q\bar{q}$ and $\pi\pi$ components,
  - Something like "correlated two-pion" state?

- Opens a promising area for new analyses of states crucial for the QCD:
  - $f_0(980)$: $qq\bar{q}\bar{q}$, $K\bar{K}$ state?,
  - $f_0(1500)$: Lowest $gg$ state? - Look at lattice QCD predictions,

- Should help end the debate about the existence of the $f_0(1370)$,

- It should help in precise determination of the CKM matrix elements and in the fight against the isobar model and old habits related with resonances.
Conclusions

- input amplitudes for the $S$, $P$, $D$ and $F$ waves constructed, at the beginning, only by fit to the data,
- just simple polynomials in energy$^2$,
- no assumption on the low threshold parameters (we use NA48/2 data),
- set of dispersion relations used in the analysis:
  - once subtracted dispersion relations ($GKPY$),
  - twice subtracted dispersion relations ($Roy$),
  - Forward Dispersion Relations ($FDR$),
  - Olsson sum rule ($SR$),
- phenomenological input partial amplitudes used up to 1420 MeV,
- above 1420 MeV - Regge parameterizations
- due to works on once and twice subtracted dispersion relations with imposed crossing symmetry condition we have in disposal very efficient set of rules for testing the partial $\pi\pi$ amplitudes in the $S$, $P$, $D$ and $F$ waves,
- we also have set of model independent unitary $\pi\pi$ amplitudes in those waves in the range from $2m_\pi$ to several GeV fulfilling very well crossing symmetry below $\sim 1100$ MeV,
- as an artefact we got very precise values of parameters for the $f_0(500)$ and $f_0(980)$ resonances